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Perturbative and Non-Perturbative Analysis of the 3'rd Order Zero Modes in the Kraichnan Model for Turbulent Advection

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The anomalous scaling behavior of the n-th order correlation functions $\mathcal{F}_n$ of the Kraichnan model of turbulent passive scalar advection is believed to be dominated by the homogeneous solutions (zero-modes) of the Kraichnan equation $\hat{\mathcal{B}}_n\mathcal{F}_n = 0$. In this paper we present an extensive analysis of the simplest (non-trivial) case of $n = 3$ in the isotropic sector. The main parameter of the model, denoted as ζ_h, characterizes the eddy diffusivity and can take values in the interval $0 \le \zeta_h \le 2$. After choosing appropriate variables we can present computer-assisted non-perturbative calculations of the zero modes in a projective two dimensional circle. In this presentation it is also very easy to perform perturbative calculations of the scaling exponent ζ_3 of the zero modes in the limit $\zeta_h \to 0$, and we display quantitative agreement with the non-perturbative calculations in this limit. Another interesting limit is $\zeta_h \to 2$. This second limit is singular, and calls for a study of a boundary layer using techniques of singular perturbation theory. Our analysis of this limit shows that the scaling exponent ζ_3 vanishes like $\sqrt{\zeta_2/\log\zeta_2}$. In this limit as well, perturbative calculations are consistent with the non-perturbative calculations.

I. INTRODUCTION

The Kraichnan model of turbulent passive scalar advection [1] pertains to a field $T(\boldsymbol{r},t)$ which satisfies the equation of motion

$$\frac{\partial T(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla T(\boldsymbol{r},t) = \kappa\nabla^2 T(\boldsymbol{r},t) + \xi(\boldsymbol{r},t). \quad (1)$$

Here $\xi(\boldsymbol{r},t)$ is a Gaussian white random force, κ is the molecular diffusivity and the driving field $\boldsymbol{u}(\boldsymbol{r},t)$ is chosen to have Gaussian statistics, and to be "rapidly varying" in the sense that its time correlation function is proportional to $\delta(t)$. The statistical quantities that one is interested in are the many point correlation functions

$$\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{2n}) \equiv \langle\langle T(\boldsymbol{r}_1,t)T(\boldsymbol{r}_2,t)\dots T(\boldsymbol{r}_{2n},t)\rangle\rangle, \quad (2)$$

where double pointed brackets denote an ensemble average with respect to the (stationary) statistics of the forcing *and* the statistics of the velocity field. Assuming that these correlation functions are scale invariant one is interested in the scaling (or homogeneity) exponent ζ_{2n} of $\mathcal{F}_{2n}$ which is defined by

$$\mathcal{F}_{2n}(\lambda\boldsymbol{r}_1,\lambda\boldsymbol{r}_2\dots\lambda\boldsymbol{r}_{2n}) = \lambda^{\zeta_{2n}}\mathcal{F}_{2n}(\boldsymbol{r}_1,\boldsymbol{r}_2\dots\boldsymbol{r}_{2n}) \ . \quad (3)$$

One expects such a scale invariant solution to exist in the inertial range, i.e. when all the separations r_{ij} satisfy $\eta \ll r_{ij} \ll L$ where η and L are the inner and outer scales respectively. It is known [1] that for $\mathcal{F}_2$ such a solution exists with $\zeta_2 = 2-\zeta_h$, where ζ_h is the exponent of the eddy-diffusivity, see Eq. (6).

The Kraichnan model is unique in the field of turbulence in that it allows the derivation [2] of an exact differential equation for this correlation function,

$$\Big[-\kappa\sum_\alpha \nabla^2_\alpha + \hat{\mathcal{B}}_{2n}\Big]\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{2n}) = \text{RHS}. \quad (4)$$

The operator $\hat{\mathcal{B}}_{2n} \equiv \sum_{\alpha>\beta}^{2n}\hat{\mathcal{B}}_{\alpha\beta}$, and $\hat{\mathcal{B}}_{\alpha\beta}$ are defined by

$$\hat{\mathcal{B}}_{\alpha\beta} \equiv \hat{\mathcal{B}}(\mathbf{r}_\alpha,\mathbf{r}_\beta) = h_{ij}(\mathbf{r}_\alpha-\mathbf{r}_\beta)\partial^2/\partial r_{\alpha,i}\partial r_{\beta,j} \ , \quad (5)$$

where the "eddy-diffusivity" tensor $h_{ij}(\boldsymbol{R})$ is given by

$$h_{ij}(\boldsymbol{R}) = h(R)[(\zeta_h+d-1)\delta_{ij} - \zeta_h R_iR_j/R^2] \ , \quad (6)$$

and $h(R) = H(R/\mathcal{L})^{\zeta_h}$. Here $\mathcal{L}$ is some characteristic outer scale of the driving velocity field. The parameter that can be varied in this model is the scaling exponent ζ_h; it characterizes the R dependence of $h_{ij}(\boldsymbol{R})$ and it can take values in the interval $[0,2]$. Finally, the RHS in Eq. (4) is known explicitly, but is not needed here. The reason is that it was argued that the solutions of this equation for $n>1$ are dominated by the homogeneous solutions ("zero-modes"), in the sense that deep in the inertial interval the inhomogeneous solutions are negligible compared to the homogeneous one. Also, it was claimed that in the inertial interval one can neglect the Laplacian operators in Eq. (4), and remain with the simpler homogeneous equations $\hat{\mathcal{B}}_{2n}\mathcal{F}_{2n} = 0$.

Exact solutions of these homogeneous equations are not easy; even in the simplest case of $n=2$ the function $\mathcal{F}_4$ depends on six independent variables (for dimensions $d>2$), and one faces a formidable analytic difficulty for exact solutions. Accordingly, several groups considered perturbative solutions in some small parameter, like ζ_h [3] or the inverse dimensionality $1/d$ [4]. The rationale

for this approach is that at $\zeta_h = 0$ and $d \to \infty$ one expects "simple scaling" with $\zeta_{2n} = n\zeta_2$. The exponent ζ_4, and later also the set ζ_{2n}, were computed as a function of ζ_h near these simple scaling limits. The other limit of $\zeta_h \to 2$ invites perturbation analysis as well, since one expects that at $\zeta_h = 2$ all the scaling exponents ζ_{2n} would vanish. Such a perturbation theory turned out to be elusive.

Recently we reported [5] that it is possible to solve exactly, eigenfunctions included, the homogeneous equation satisfied by the 3'rd order correlation function $\mathcal{F}_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$. Note that in Kraichnan's model all the odd-order correlation functions $\mathcal{F}_{2n+1}$ are zero because of symmetry under the transformation $T \to -T$. This symmetry disappears for example [6] if the random force $\xi(\boldsymbol{r}, t)$ is not Gaussian (but δ-correlated in time), and in particular if it has a non-zero third order correlation

$$\mathcal{D}_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) \equiv \int dt_1 dt_2 \langle \xi(\boldsymbol{r}_1, t_1)\xi(\boldsymbol{r}_2, t_2)\xi(\boldsymbol{r}_3, 0)\rangle. \quad (7)$$

With such a forcing the third order correlator is non-zero, and it satisfies the equation

$$\hat{\mathcal{B}}_3 \mathcal{F}_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = \mathcal{D}_3 \,, \quad \hat{\mathcal{B}}_3 \equiv \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{23} \ . \quad (8)$$

This equation pertains to the inertial interval and accordingly we neglected the Laplacian operators. We also denoted $\mathcal{D}_3 = \lim_{\boldsymbol{r}_{\alpha\beta}\to 0} \mathcal{D}_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$. The solution of this equation is a sum of inhomogeneous and homogeneous contributions, and below we study the latter. We will focus on scale invariant homogeneous solutions which satisfy $\mathcal{F}_3(\lambda\boldsymbol{r}_1, \lambda\boldsymbol{r}_2, \lambda\boldsymbol{r}_3) = \lambda^{\zeta_3}\mathcal{F}_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$. We refer to these as the "zero modes in the scale invariant sector". We note that the scaling exponent of the *inhomogeneous* scale invariant contribution can be read directly from power counting in Eq. (8) (leading to $\zeta_3 = \zeta_2$). Any different scaling exponent can arise only from homogeneous solutions that do not need to balance the constant RHS. The scale invariant solutions of Eq. (8) live in a projective space whose dimension is lowered by unity compared to the most general form. These solutions do not depend on three separations but rather on two dimensionless variables that are identified below. It will be demonstrated how boundary conditions arise in this space for which the operator $\hat{\mathcal{B}}_3$ is neither positive nor self-adjoint.

In Section 2 we present the transformation of variables that leads to a precise identification of the projective space. In this space we present the differential equation that needs to be studied, and derive the boundary conditions in the projective space. In section 3 we discuss the perturbation theory that leads to the solution of the scaling exponents of the zero modes in the limit $\zeta_h \to 0$. It is shown that the choice of coordinates of section 2 leads to a particularly transparent theory in this limit. In section 4 we present the perturbation theory in the limit $\zeta_h \to 2$. It turns out that this is a singular perturbation theory, and we discuss the analytic matchings across boundary layers and near the "fusion singularity" which are required to understand this limit, leading to a non-analytic dependence of ζ_3 on ζ_2. In Section 5 we deal with the non-perturbative calculation, culminating with solutions of ζ_3 as a function of ζ_h throughout the range $0 \le \zeta_h \le 2$. It is demonstrated that the non-perturbative solutions are in agreement with the perturbative calculations at the two ends of this interval. Section 6 is devoted to a summary and a discussion.

II. TRANSFORMATION OF VARIABLES

In this section we describe the transformation of variables in the operator $\mathcal{B}_3$ to new variables that are denoted below as s, ρ, ϕ. We first note that equation (8) is invariant under space translation, under the action of the d dimensional rotation group SO(d), and under permutations of the three coordinates. Accordingly, we may seek solutions in the scalar representation of SO(d), where the solution depends on the three separations r_{12}, r_{23} and r_{31} only. In the first stage we transform coordinates to the variables $x_1 = |\boldsymbol{r}_2 - \boldsymbol{r}_3|^2$, $x_2 = |\boldsymbol{r}_3 - \boldsymbol{r}_1|^2$, $x_3 = |\boldsymbol{r}_1 - \boldsymbol{r}_2|^2$, defining

$$F_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = f_3(x_1, x_2, x_3) \ . \quad (9)$$

By the chain rule,

$$\partial_{1i} F_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = 2(r_{13i}\partial_2 f_3 + r_{12i}\partial_3) f_3(x_1, x_2, x_3), \quad (10)$$

where $\partial_{1i} \equiv (\partial/\partial_{r_{1i}})$, $\partial_2 \equiv (\partial/\partial_{x_2})$, and $r_{12i} \equiv r_{1i} - r_{2i}$ etc.

Another application of the chain rule gives

$$\begin{aligned}\partial_{2j}\partial_{1i} = {} & 4r_{13i}r_{23j}\partial_1\partial_2 + 4r_{12i}r_{23j}\partial_1\partial_3 \\ & -4r_{13i}r_{12j}\partial_2\partial_3 - 4r_{12i}r_{12j}\partial_3^2 - 2\delta_{ij}\partial_3 \ . \end{aligned} \quad (11)$$

(For brevity we display only the differential operators explicitly). Using $2\boldsymbol{r}_{12}\cdot\boldsymbol{r}_{13} = -x_1 + x_2 + x_3$, and similar identities, we can now obtain

$$\begin{aligned}\delta_{ij}\partial_{2j}\partial_{1i} = {} & 2(x_1 + x_2 - x_3)\partial_1\partial_2 + 2(-x_1 + x_2 - x_3)\partial_1\partial_3 \\ & + 2(x_1 - x_2 - x_3)\partial_2\partial_3 - 4x_3\partial_1\partial_2 - 2d\partial_3 \ , \end{aligned} \quad (12)$$

and

$$\begin{aligned}\frac{r_{12i}r_{12j}}{r_{12}^2}\partial_{2j}\partial_{1i} = {} & \frac{(x_1 - x_2 - x_3)(x_1 - x_2 + x_3)}{x_3}\partial_1\partial_2 \\ & +2(-x_1 + x_2 - x_3)\partial_1\partial_3 + 2(x_1 - x_2 - x_3)\partial_2\partial_3 \\ & - 4x_3\partial_1\partial_2 - 2\partial_3 \ . \end{aligned} \quad (13)$$

Further calculations, using (12) and (13) give

$$\mathcal{B}_{12} \equiv r_{12}^{\zeta_h}\Big[(d+\zeta_h-1)\delta_{ij} - \zeta_h \frac{r_{12i}r_{12j}}{r_{12}^2}\Big]\partial_{2j}\partial_{1i} \tag{14}$$
$$= x_3^{\zeta_h/2}\Big[(d-1)o_1 + (d-1)(d+\zeta_h)o_2 + \frac{\zeta_h}{x_3}o_3\Big] \ ,$$

where

$$o_1 = 2(x_1+x_2-x_3)\partial_1\partial_2 + 2(-x_1+x_2-x_3)\partial_1\partial_3 \tag{15}$$
$$+2(x_1-x_2-x_3)\partial_2\partial_3 - 4x_3\partial_1\partial_2\,,$$
$$o_2 = -2\partial_3\,, \tag{16}$$
$$o_3 = [x_3(x_1+x_2-x_3) \tag{17}$$
$$-(x_1-x_2-x_3)(x_1-x_2+x_3)]\partial_1\partial_2 \ .$$

The reader should note that the o_i operators do not depend on the parameters of the problem. The operators $\mathcal{B}_{23}$ and $\mathcal{B}_{31}$ are obtained from $\mathcal{B}_{12}$ by cyclic permutations of the indices, thus completing the transformation of $\mathcal{B}_3$ to the x variables.

Note that not every point in the x_1, x_2, x_3 space corresponds to a physical configuration. The triangle inequalities between the pairwise distances translates to the condition

$$2(x_1x_2 + x_2x_3 + x_3x_1) \geq x_1^2 + x_2^2 + x_3^2. \tag{18}$$

This inequality describes a circular cone in the x_1, x_2, x_3 space whose axis is the line $x_1 = x_2 = x_3$, tangent to the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. The group of permutations between the x_i axes acts very simply on this cone, corresponding to a C_6 operation.

The presence of symmetries motivate a new parameterization of the cone by three new coordinates s, ρ, ϕ:

$$x_n = s[1 - \rho\cos(\phi + \tfrac{2}{3}n\pi)]\,, \tag{19}$$
$$0 \leq s < \infty, \quad 0 \leq \rho \leq 1, \quad 0 \leq \phi \leq 2\pi \ .$$

The new space is a direct product of three intervals, and this fact will simplify the discussion of the boundary conditions. The s coordinate measures the overall scale of the triangle defined by the original $\boldsymbol{r}_i$ coordinates, and configurations of constant ρ and ϕ correspond to similar triangles. The ρ coordinate describes the deviation of the triangle from the equilateral configuration ($\rho = 0$) up to the physical limit of three collinear points attained when $\rho = 1$; ϕ does not have a simple geometric meaning. Finally we note that the variables s, ρ and $\cos(3\phi)$ are symmetric in the x_i variables. Accordingly any function of these variable is automatically invariant under the permutation of the x_i variables. We will use this property below.

The final form of the equation is achieved by transforming o_i operators to the variables s, ρ, ϕ. To this end we compute the Jacobian of the transformation (19) using *Mathematica*,

$$J \equiv \frac{\partial(s,\,\rho,\,\phi)}{\partial(x_1, x_2, x_3)} = \frac{1}{\sqrt{3}s^2\rho} \tag{20}$$
$$\times\begin{bmatrix} 1, & -\rho+\cos(\phi)+\sqrt{3}\sin(\phi), & \sqrt{3}\cos(\phi)-\sin(\phi) \\ 1, & -\rho+\cos(\phi)-\sqrt{3}\sin(\phi), & -\sqrt{3}\cos(\phi)+\sin(\phi) \\ 1, & -\rho-2\cos(\phi), & 2\sin(\phi)/\sqrt{3} \end{bmatrix}.$$

The Jacobian matrix J is substituted in the chain rule to give the transformation of the derivatives of f_3 with respect to the x variables in terms of the new variables s, ρ, ϕ. It is convenient to perform the tedious calculations using *Mathematica* ending up with expressions for o_1, o_2, and o_3.

We present the final long result as a table, in which each item is of the form

$$\{\bar{s}, \bar{\rho}, \bar{\phi},\ n,\ m,\ c\}$$

representing a term of the form

$$c\rho^n \begin{matrix}\cos\\ \sin\end{matrix}(m\phi)s^{\bar{s}}\partial_s^{\bar{s}}\partial_\rho^{\bar{\rho}}\partial_\phi^{\bar{\phi}}.$$

The trigonometric functions cos or sin appear if $\bar{\phi}$ is even or odd respectively. The o_i operators are sums of such terms. The tables listing the terms appearing in each of the operators are presented in Appendix A.

The upshot of the transformation of the linear operator $\hat{\mathcal{B}}_3$ to the new coordinates is that we derive a second order linear partial differential operator in the s, ρ, ϕ variables. At this point we take advantage of the scale invariance of the differential equation which allows to seek scale invariant solutions of the form $s^{\zeta_3/2}f(\rho,\phi)$. Acting on functions of this form, the operators $s\partial_s$ and $s^2\partial_s^2$ become scalar multiplications by $\frac{1}{2}\zeta_3$ and $\frac{1}{2}\zeta_3(\frac{1}{2}\zeta_3 - 1)$ respectively. The action of the operator $\hat{B}_3$ yields an equation for $f(\rho,\phi)$, which is the basic equation we study in this paper

$$\hat{B}_3(\zeta_3)f(\rho,\phi) = [a(\rho,\phi)\partial_\rho^2 + b(\rho,\phi)\partial_\phi^2 + c(\rho,\phi)\partial_\rho\partial_\phi \tag{21}$$
$$+u(\rho,\phi,\zeta_3)\partial_\rho + v(\rho,\phi,\zeta_3)\partial_\phi + w(\rho,\phi,\zeta_3)\big]f(\rho,\phi) = 0 \ .$$

We note that Eq. (21) can be written in a coordinate-free form as

$$\big[-\boldsymbol{\nabla}\cdot\overleftrightarrow{P}(\rho,\phi)\cdot\boldsymbol{\nabla} + \boldsymbol{q}(\rho,\phi,\zeta_3)\cdot\boldsymbol{\nabla} \tag{22}$$
$$+ w(\rho,\phi,\zeta_3)]f(\rho,\phi) = 0\,,$$

where $\boldsymbol{\nabla}$ is the gradient operator in the ρ, ϕ space. The identification of the tensor $\overleftrightarrow{P}$ and the row vector $\boldsymbol{q}$ is obtained by comparison with the explicit form (21). The new operator $\hat{B}_3$ depends on ζ_3 as a parameter and it acts on the unit circle described by the polar ρ, ϕ coordinates. The circle represents the projective space of the physical cone described above. We will see that the availability of a compact domain (the projective space) will lead to the existence of a discrete spectrum of the zero modes.

The discrete permutation symmetry of the original Eq. (8) results in a symmetry of Eq. (21) with respect

to the 6 element group generated by the transformation $\phi \to \phi + 2\pi/3$ (cyclic permutation of the coordinates in the physical space) and $\phi \to -\phi$ (exchange of coordinates). This symmetry extends to a full $U(1)$ symmetry in the two marginal cases of $\zeta_h = 0$ and $\zeta_h = 2$ (see [7,8] for a discussion of the latter limit) for which all the coefficients in (21) become ϕ-independent. The coefficients in (21) all have a similar structure, and for example $a(\rho, \phi)$ reads

$$a(\rho,\phi) = \sum_n [1-\rho\cos(\phi+\tfrac{2}{3}\pi n)]^{(\zeta_h-2)/2}\tilde{a}(\rho,\phi+\tfrac{2}{3}\pi n)\,,$$

where $\tilde{a}(\rho,\phi)$ is a low order polynomial in ρ, $\cos\phi$ and $\sin\phi$ which vanishes at $\rho = 1, \phi = 0$. We see that the coefficients are analytic everywhere on the circle except at the three points $\rho = 1$, $\phi = 2\pi n/3$ where $n = 0, 1, 2$. These points correspond to the fusion of one pair of coordinates, and the coefficients exhibit a branch point singularity there. This singularity is a consequence of the non-analyticity of the driving velocity field whose eddy-diffusivity is therefore characterized by a non-integer exponent. This singularity leads to a nontrivial asymptotic behavior of the solutions which had been described before in terms of the fusion rules [9,10]. Since the coefficient $\tilde{a}$ vanishes at the fusion point, the singularity is always sub-leading with respect to terms that come from non-fusing coordinates. Indeed, in [9,10] it was explained that exposing the singularity calls for taking a derivative with respect to the fusing coordinates. Note that for $\zeta_h = 2$ the singularity disappears trivially. For $\zeta_h = 0$ there is also no singularity since $\tilde{a}$ exactly compensates for the inverse power.

The boundary conditions follow naturally when one realizes that $\hat{B}_3$ is elliptic for points strictly inside the physical circle. In the presentation of Eq. (22) this means that

$$\det(\overleftrightarrow{P}) > 0 \quad \text{for } \rho < 1. \tag{23}$$

This property is a consequence of the ellipticity of the original operator $\hat{\mathcal{B}}_3$. On the other hand $\hat{B}_3$ becomes singular on the boundary $\rho = 1$, where the coefficients $a(\rho, \phi)$ and $c(\rho, \phi)$ vanish. In other words,

$$\overleftrightarrow{P} \cdot \boldsymbol{n}|_{\rho=1} = 0, \tag{24}$$

where $\boldsymbol{n}$ is a unit vector normal to the boundary. This singularity reflects the fact that this is the boundary of the physical region. It follows that $\hat{B}_3$ restricted to the boundary becomes a relation between the function $f(\rho{=}1, \phi) \equiv g(\phi)$ and its normal derivative $\partial_\rho f(\rho, \phi)|_{\rho=1} \equiv h(\phi)$. The relation is

$$bg''(1,\phi) + uh(1,\phi) + vg'(1,\phi) + wg(1,\phi) = 0\ . \tag{25}$$

Solutions of Eq. (21) which do not satisfy this boundary condition are singular, with infinite ρ derivatives at $\rho = 1$. Such solutions are not physical since they involve infinite correlations between the dissipation (second derivative of the field) and the field itself when the geometry becomes collinear, but without fusion.

It is important to stress that the reason for the regularity of the solutions that do satisfy the boundary conditions is that $\det \overleftrightarrow{P}$ vanishes as a simple zero near the boundary, i.e. like $(1-\rho)$. Consequently it is possible to find solutions that behave like $(1-\rho)^0 \sim O(1)$ near the boundary.

Finally, we are facing the problem of solving Eq. (21) together with the boundary condition (25). This homogeneous boundary value problem always have a trivial solution $f = 0$. The condition for the existence of a non-trivial solution is that $\hat{B}_3$ is not invertible. We expect that non-invertibility will not be a generic phenomenon for an arbitrary value of ζ_3. Finding the set of ζ_3 for which $\hat{B}_3$ is not invertible becomes a generalized eigenvalue problem with the exponents ζ_3 playing the role of eigenvalues. We first discuss this program in the limit of small ζ_h.

III. PERTURBATION THEORY NEAR $\zeta_H = 0$

The shape of our compact domain invites a Fourier representation in ϕ for the function $f(\rho, \phi)$. The permutation symmetry implies that only cos will appear in this representation, and the index will be divisible by 3. The general equation (21) will then mix different Fourier modes. On the other hand, in the limit $\zeta_h = 0$ the situation simplifies considerably.

A. Zero Modes at $\zeta_h = 0$

The differential equation reduces in the case $\zeta_h = 0$ to the simple form

$$[\rho^2(1-\rho^2)\partial_\rho^2 + \rho(1-d\rho^2)\partial_\rho + \partial_\phi^2 + \lambda\rho^2]f(\rho,\phi) = 0, \tag{26}$$

where

$$\lambda \equiv \tfrac{1}{2}\zeta_3(\tfrac{1}{2}\zeta_3 + d - 1)\ . \tag{27}$$

Since the coefficients of Eq. (26) are independent of ϕ the Fourier modes are decoupled, and we may seek solutions of the form $f_m(\rho)\cos m\phi$ with m divisible by 3. The functions f_m obey the ODEs

$$[\rho^2(1-\rho^2)\partial_\rho^2 + \rho(1-d\rho^2)\partial_\rho - m^2 + \lambda\rho^2]f_m(\rho) = 0. \tag{28}$$

Examining the resulting differential equation we note that the coefficient of the highest derivative in ρ vanishes as a double zero at $\rho = 0$ and as a single zero at

$\rho = 1$. Since the order of the zero is not more than the order of the derivative the Frobenius theory [11] of regular singularities is applicable to both boundaries. Namely, the complete family of solutions in the vicinity of a given singular point ρ_0 is spanned by functions that can be represented as

$$f_m^{(i)}(\rho) = (\rho - \rho_0)^{z_i} \log(\rho - \rho_0)^{k_i} \sum_{p=0}^{\infty} a_{i,p}(\rho - \rho_0)^p,$$
$$i = 1, 2, \quad k_i = 0 \text{ or } 1 \ , \tag{29}$$

where the sum is convergent in a neighborhood of ρ_0. One of the indices k_i is always zero. When the indices $z_{1,2}$ are different, we chose arbitrarily $z_1 > z_2$ and then k_1 is zero. When $|z_1 - z_2|$ is not an integer one also has $k_2 = 0$. In cases in which the indices z_i coincide we will chose $k_1 = 0$, and $k_2 = 1$. The numerical values of the indices are obtained by substituting a solution $(\rho - \rho_0)^z$ in the differential equation, collecting the coefficients of the leading terms near the singularity, and equating to zero. We refer to the indices z_i as the "Frobenius exponents". Eq. (28) has regular singularities at both boundaries $\rho = 0$, 1, with Frobenius exponent sets of $-m$, m and 0, $(3-d)/2$ respectively. The singularity at $\rho = 1$ arises from the singularity of the original PDE (21) at this boundary, and the boundary condition picks the regular solution, *i.e.,* $f_m(\rho) \sim (1-\rho)^0$ as $\rho \to 1$. On the other hand, the singularity at $\rho = 0$ is an artifact of the transformation to polar coordinates; however, analyticity of the solution at $\rho = 0$ requires that $f_m(\rho) \sim \rho^m$ as $\rho \to 0$, specifying the second boundary condition for (28).

To proceed with the solution of Eq. (28) we make the transformations $\tilde{f}_m(\tilde{\rho}) = \rho^m f_m(\rho)$ and $\tilde{\rho} = \rho^2$, and obtain the hypergeometric equation for $\tilde{f}_m$,

$$\big\{(1-\tilde{\rho})\tilde{\rho}\partial^2_{\tilde{\rho}} + [m + 1 - (m + \tfrac{1}{2}(d+1)]\tilde{\rho})\partial_{\tilde{\rho}} + \tfrac{1}{4}[\lambda - m(m-1+d)]\big\}\tilde{f}_m(\tilde{\rho}) = 0 \ . \tag{30}$$

This equation is standard, see [12]. The unique solution of Eq. (30) which satisfies the boundary condition at 0 (up to a multiplication by a constant) is

$$\tilde{f}_m(\tilde{\rho}) = \ _2F_1(a, b; m+1; \tilde{\rho}), \tag{31}$$

where

$$a = \tfrac{1}{2}(m - \tfrac{1}{2}\zeta_3) \ , \qquad b = \tfrac{1}{2}(m + d - 1 + \tfrac{1}{2}\zeta_3) \ . \tag{32}$$

It follows from the theory of the hypergeometric functions that the function $\tilde{f}_m$ defined in (31) is regular at $\rho = 1$ only if either a or b equal $-n$, where n is a nonnegative integer. In such a case $\tilde{f}_m$ becomes a polynomial of degree n. The spectrum of ζ_3 now follows from (32) and consists of two sets,

$$\zeta^+_{3(m,n)} = 2(m + 2n), \qquad \zeta^-_{3(m,n)} = -2(d - 1 + m + 2n). \tag{33}$$

Since $\lambda(\zeta^+_{3(m,n)}) = \lambda(\zeta^-_{3(m,n)})$, [cf. Eq. (27)], the corresponding eigenfunctions depend only on n and m. Expressed in terms of the original variables the eigenfunctions are

$$f_{(m,n)}(\rho,\phi) = \rho^m \ _2F_1(-n, n + m + (d-1)/2; m+1; \rho^2)\cos(m\phi) \ . \tag{34}$$

The zero-modes of the first set, with positive values of ζ_3 are the ones to be matched at the outer scale. Of these, the most relevant non-trivial zero mode is $f_{0,1}$ with $\zeta^+_{3(0,1)} = 4$, still less relevant than the scaling of the forced solution.

B. Zero Modes for $0 < \zeta_h \ll 1$

Perturbation theory for $\zeta_h \to 0$ will be carried out by expanding the solution for small ζ_h in terms of the $\zeta_h = 0$ eigenfunctions $f_{(m,n)}$, in a procedure very similar to time independent perturbation theory in quantum mechanics. We first cast Eq. (28) in Sturm-Liouville form

$$\Big\{\frac{1}{\rho(1-\rho^2)^{(d-3)/2}}\partial_\rho\Big[\rho(1-\rho^2)^{(d-1)/2}\partial_\rho\Big] - \frac{m^2}{\rho^2}\Big\} f_m(\rho) = -\lambda f_m(\rho) \ . \tag{35}$$

It follows that the zero-modes $f_{(m,n)}$ form an orthogonal set with respect to the inner product

$$\langle f, g\rangle_0 \equiv \int_{\rho\le 1} \frac{d\rho d\phi}{2\pi}\rho(1-\rho^2)^{(d-3)/2} f(\rho,\phi)g(\rho,\phi) \ . \tag{36}$$

We now assume that we may expand the eigenfunction and eigenvalues as

$$f(\rho,\phi) = f_{(m,n)}(\rho,\phi) + \zeta_h f^{(1)}(\rho,\phi) + O(\zeta_h^2) \ , \tag{37}$$
$$\zeta_3 = \zeta_{3(m,n)} + \zeta_3^{(1)} + O(\zeta_h^2) \ ; \tag{38}$$

writing

$$B(\zeta_h,\zeta_3) = B(0,\zeta_{3(m,n)}) + \zeta_h\Big[\partial_{\zeta_h} B(0,\zeta_{3(m,n)}) + \zeta_3^{(1)}\partial_{\zeta_3}B(0,\zeta_{3(m,n)})\Big] + O(\zeta_h^2) \ , \tag{39}$$

Eq (21) becomes, to order ζ_h,

$$B(0,\zeta_{3(m,n)})f^{(1)}(\rho,\phi) + \Big[\partial_{\zeta_h}B(0,\zeta_{3(m,n)}) + \zeta_3^{(1)}\partial_{\zeta_3}B(0,\zeta_{3(m,n)})\Big] f_{(m,n)}(\rho,\phi) = 0 \ . \tag{40}$$

Taking the inner product $\langle\rangle_0$ of Eq. (40) with $f_{(m,n)}$, and using the self-adjointness of $B(0,\zeta_{3(m,n)})$ with respect to this inner product, there follows an expression for the first correction to ζ_3,

$$\zeta_3^{(1)} = -\frac{\big\langle f_{(m,n)}, \partial_{\zeta_h}B(0,\zeta_{3(m,n)})f_{(m,n)}\big\rangle_0}{\big\langle f_{(m,n)}, \partial_{\zeta_3}B(0,\zeta_{3(m,n)})f_{(m,n)}\big\rangle_0} \ . \tag{41}$$

The integrals in (41) were programmed using *Mathematica*, and we used the program to generate expressions for the first few values for $\zeta_3^{(1)}$ presented below:

m	n	$\zeta^+_{3(m,n)}$	$\zeta_3^{+(1)}$	$\zeta^-_{3(m,n)}$	$\zeta_3^{-(1)}$
0	0	0	0	$-2d+2$	-1
0	1	4	$\frac{2(2-d)}{d-1}$	$-2d-2$	$\frac{3-d}{d-1}$
0	2	8	$\frac{2\left(159+91\,d-31\,d^2-19\,d^3-2\,d^4\right)}{-48+4\,d+32\,d^2+11\,d^3+d^4}$	$-2d-6$	$\frac{-3\left(90+62\,d-10\,d^2-9\,d^3-d^4\right)}{-48+4\,d+32\,d^2+11\,d^3+d^4}$
0	3	12	$\frac{6\left(780+503\,d-89\,d^2-103\,d^3-19\,d^4-d^5\right)}{-480-8\,d+324\,d^2+142\,d^3+21\,d^4+d^5}$	$-2d-10$	$-\frac{4200+3010\,d-210\,d^2-476\,d^3-93\,d^4-5\,d^5}{-480-8\,d+324\,d^2+142\,d^3+21}$
3	0	6	$\frac{-3\left(15+8\,d+d^2\right)}{-2+d+d^2}$	$-2d-4$	$-\frac{(5+d)\,(7+2\,d)}{2-d-d^2}$
3	1	10	$-\frac{(9+d)\left(-489-131\,d+136\,d^2+53\,d^3+5\,d^4\right)}{-144-36\,d+100\,d^2+65\,d^3+14\,d^4+d^5}$	$-2d-8$	$-\frac{(9+d)\left(441+135\,d-104\,d^2-42\,d^3-4\,d^4\right)}{-144-36\,d+100\,d^2+65\,d^3+14\,d^4+d^5}$
6	0	12	$\frac{3\,(3+d)\,(11+d)^2\left(370+216\,d+37\,d^2+2\,d^3\right)}{3840+544\,d-2584\,d^2-1460\,d^3-310\,d^4-29\,d^5-d^6}$	$-2d-10$	$\frac{5\,(11+d)\left(6558+6508\,d+2409\,d^2+412\,d^3+33\,d^4+d^5\right)}{-3840-544\,d+2584\,d^2+1460\,d^3+310\,d^4+29\,d^5+d^6}$

(42)

IV. PERTURBATION THEORY IN THE LIMIT $\zeta_H \to 2$

In this section we treat the case $\zeta_2 = 2 - \zeta_h \ll 1$ perturbatively. The perturbation theory is singular, since the leading order approximation, obtained by setting the small parameter $\zeta_2 = 0$ is not valid throughout the entire domain. We employ boundary layer techniques and matching of asymptotics to obtain an asymptotic approximation for ζ_3 in this limit. As expected, ζ_3 goes to 0 with ζ_2, but with a non-trivial dependence

$$\zeta_3 = O\big(\sqrt{\zeta_2/\log\zeta_2}\big) \ . \tag{43}$$

A. Leading order solution

As in the case $\zeta_h = 0$ discussed above, substituting $\zeta_h = 2$ into the zero-mode equation (21) yields an equation with coefficients which are independent of the variable ϕ. This equation reads

$$\begin{aligned}&[\rho^2(1-\rho^2)^2\partial_\rho^2 + \rho(1-\rho^2)(1-2\rho^2+\zeta_3\rho^2)\partial_\rho \qquad (44)\\ &+(1-\rho^2)\partial_\phi^2 + w(\rho)]f(\rho,\phi) = 0\,,\\ &w(\rho) = \rho^2\frac{\zeta_3}{2d}\Big[(d+2)(d-1) + (\frac{\zeta_3}{2}-1)\Big((d-1)(\rho^2+1)\\ &+\rho^2-1\Big)\Big] \ . \qquad (45)\end{aligned}$$

This significant simplification is a consequence of the higher symmetry of the passive scalar equation in this limit, as discussed in detail in Ref . [7,8].

Making use of the symmetry, we look for solutions of the form

$$f(\rho,\phi) = f_m(\rho)\cos m\phi \quad m = 3n,\, n = 0,1,2\ldots \tag{46}$$

The functions f_m satisfy the ODEs

$$\begin{aligned}&[\rho^2(1-\rho^2)^2\partial_\rho^2 + \rho(1-\rho^2)(1-2\rho^2+\zeta_3\rho^2)\partial_\rho\\ &-m^2(1-\rho^2) + w(\rho)]f_m(\rho) = 0 \ .\end{aligned} \tag{47}$$

Equation (47) has regular singularities at the points 0 and 1. In close analogy with the case $\zeta_h = 0$ we find that the Frobenius exponents at 0 are $\pm m$, and choose the solution which behave as ρ^m at 0, thus providing the boundary condition at 0.

The behavior near the boundary $\rho = 1$ is different. In contrast with the case $0 \le \zeta_h < 2$ where one of the Frobenius exponents is always 0, when $\zeta_h = 2$ the Frobenius exponents depend on the value of ζ_3, and generally neither is 0. Thus, the qualitative behavior of the zero modes for $0 < \zeta_2 = 2 - \zeta_h \ll 1$ near $\rho = 1$ is different from that of the solutions of Eq. (45) where ζ_h is set equal to 2. In other words, we expect the existence of a boundary layer near $\rho = 1$; Eq. (45) describes well only the behavior of the *outer* solution, namely the leading order approximation away from the boundary layer.

The outer solution may now be written explicitly. Using standard transformations [12] Eq. (47) may be reduced to a hypergeometric equation, whose solution which obeys the $\rho = 0$ boundary condition is

$$f_m(\rho) = \rho^m \ {}_2F_1(m/2,(m+1)/2;m+1;\rho^2) \ . \tag{48}$$

Since we anticipate that $\zeta_3 \ll 1$, we give in (48) the solution of equation (47) taking $\zeta_3 = 0$. This approximation is justified to leading order *a-posteriori*, when we find that ζ_3 is indeed small. The outer solution is constructed from the f_m's by

$$f^{out}(\rho,\phi) = \sum_{m=0,3,6,\ldots} \nu_m f_m(\rho)\cos m\phi \tag{49}$$

The coefficient ν_m are unknown at this stage and they will be determined by matching with the inner solution.

The outer solution is a valid asymptotic approximation of the true solution only when $\zeta_2 \ll 1-\rho$. Since we expect the solution to vary very rapidly within a boundary layer near $\rho = 1$, the inner solution, valid within the boundary layer, should be expressed in terms of a "fast" variable τ that changes on a scale inversely proportional to ζ_2, namely,

$$\rho = 1 + \zeta_2\tau. \tag{50}$$

We therefore change the variable ρ to τ in the differential equation (21) and keep the leading terms in ζ_2. The resulting equation for the inner solution is

$$\{\tau[\tau - \frac{d+1}{2d}p(\phi)]\partial_\tau^2 + (\frac{(d+3)(d-1)}{4d}p(\phi) - \frac{1}{2}\tau)\partial_\tau + w(1)/4\} f^{\rm in}(\tau,\phi) = 0 \ , \quad (51)$$

where

$$p(\phi) = \sum_{\phi'=\phi,\,\phi\pm 2\pi/3} \log(1-\cos\phi')(\cos\phi' - \cos 2\phi'), \quad (52)$$

and the function w is defined in (45). It is crucial for the matching between inner and outer solution to realize that Eq. (51) is not valid near the fusion points $\phi = 2\pi n/3$. The significance of this fact is explained below.

Equation (51) has a regular singularity at $\tau = 0$ which corresponds to the $\rho = 1$ boundary, and as in the other cases demanding regularity gives a boundary condition at this point. The equation can be again transformed into the hypergeometric equations, so it has solutions of the form

$$f^{\rm in}(\tau,\phi) = \mu(\phi) \times {}_2F_1\left(-\frac{(d-1)\zeta_3}{4}, -\frac{1}{2}; \frac{(d+3)(d-1)}{2(d+1)}; \frac{2d}{d+1}\frac{\tau}{p(\phi)}\right) \ . \quad (53)$$

Here again we used the smallness of ζ_3 and neglected terms of higher order in it. The function $f^{\rm in}$ provides an asymptotic approximation to the actual solution when $-\tau \ll (1/\zeta_2)$. It depends on the function $\mu(\phi)$ that will be determined by matching to the outer solution.

B. Asymptotic matching

The next step involves matching of the two approximations $f^{\rm in}$ and $f^{\rm out}$ in their common region of validity

$$\zeta_2 \ll 1-\rho \ll 1 \ . \quad (54)$$

We perform the matching using standard boundary layer methods by examining the asymptotic behavior of $f^{\rm in}$ and $f^{\rm out}$ in the matching region (54) and balancing coefficients. We make use of the asymptotic behavior of the hypergeometric functions [12]. For the outer solution we need the asymptotics of the hypergeometric function with argument close to 1, yielding

$$f^{\rm out}(\rho,\phi) \underset{1-\rho\ll 1}{\sim} \sum_m 2^m \nu_m \left[1 - m\sqrt{2(1-\rho)}\right]\cos(m\phi) \ . \quad (55)$$

For the inner solution we use the behavior of the hypergeometric function for large negative values, and obtain, using the relation between ρ and τ

$$f^{\rm in}(\rho,\phi) \underset{1-\rho\ll 1}{\sim} \mu(\phi)\left[1 + c\zeta_3\sqrt{\frac{1-\rho}{\zeta_2 p(\phi)}}\,\right] , \quad (56)$$

where

$$c = \frac{\sqrt{\pi}\,\Gamma\left[\frac{(d+3)(d-1)}{2(d+1)}\right]}{\Gamma\left[\frac{(d+3)(d-1)}{2(d+1)} + \frac{1}{2}\right]}\frac{d-1}{4}\sqrt{\frac{d+1}{2d}}. \quad (57)$$

To proceed with the matching of (55) and (56), we first match the $O(1)$ terms [for small $(1-\rho)$], giving

$$\mu(\phi) = \sum_m 2^m \nu_m \cos(m\phi), \quad (58)$$

i.e. $2^m\nu_m$ are just the coefficients of the Fourier series of $\mu(\phi)$. Next we have to match the $O(\sqrt{1-\rho}\,)$ terms giving

$$\frac{c\zeta_3}{\sqrt{\zeta_2 p(\phi)}}\mu(\phi) = \sum_m -m\sqrt{2}\; 2^m \nu_m \cos(m\phi) \ . \quad (59)$$

We expand $p(\phi)^{-1/2}$ in Fourier series,

$$p(\phi)^{-1/2} = \sum_m p_m \cos(m\phi) \quad (60)$$

and substitute in (59) to get, using (58)

$$\sum_{mn} \frac{c\zeta_3}{2\sqrt{\zeta_2}} p_n \nu_m (\cos[(n-m)\phi] + \cos[(n+m)\phi]) = \sum_m -m\sqrt{2}\nu_m \cos(m\phi) \ . \quad (61)$$

Equating Fourier coefficients of the same order yields finally

$$\sum_n \frac{c\zeta_3}{2\sqrt{\zeta_2}} p_n (\nu_{m+n} + \nu_{|m-n|}) = -m\nu_m. \quad (62)$$

The matching condition (62) is a generalized eigenproblem for the infinite vector ν_m, with eigenvalues ζ_3.

Since ζ_3 appears in (62) only through the combination $\zeta_3/\sqrt{\zeta_2}$, one would be led to conclude that $\zeta_3 = O(\sqrt{\zeta_2}\,)$ for all non-trivial solutions of (62) with different numerical coefficients. This consideration is modified, however, since the coefficients p_n themselves also depend on ζ_2 as will be now demonstrated.

It follows from the definition of $p(\phi)$ (52), that

$$p(\phi) = O[\phi^2 \log(\phi)] \quad (63)$$

for ϕ small. Note that the Fourier coefficients p_n are written as integrals

$$p_n = \frac{1}{\pi}\int_0^{2\pi} \frac{d\phi}{\sqrt{p(\phi)}} \quad \text{for} \ \ n > 1 \ . \quad (64)$$

All these integrals diverge at $\phi = 0$, which is precisely the fusion point where the approximation (53) ceases to

be valid, requiring a more careful examination of the behavior in this region. In deriving Eq. (53) we made the approximation

$$(1-\rho\cos\phi)^{-\zeta_2/2} \sim 1-\tfrac{1}{2}\zeta_2\log(1-\rho\cos\phi) \sim 1-\tfrac{1}{2}\zeta_2\log(1-\cos\phi)\ , \qquad (65)$$

for $1-\rho = O(\zeta_2)$. When $\phi^2 = O(\zeta_2)$ the second approximation is no longer valid, and one has instead

$$(1-\rho\cos\phi)^{-\zeta_2/2} \sim 1-\tfrac{1}{2}\zeta_2\log[(1-\rho)+\tfrac{1}{2}\phi^2]\,, \qquad (66)$$

so that instead of (51) we get a similar equation in which $p(\phi)$ is replaced by $\log(-\zeta_2\tau+\frac{1}{2}\phi^2)$. The resulting equation is no longer integrable in terms of hypergeometric functions. However, we still expect the asymptotic approximation (56) to be valid, but for small ϕ the function $p(\phi)$ is no longer given by (52). We may estimate $p(\phi)$ for small ϕ by the following consideration. Examining Eq. (51) we see that $p(\phi)$ is the value of τ where the coefficient of the second derivative crosses over from quadratic behavior to linear behavior, and the coefficient of the first derivative crosses over from linear to constant behavior. When $\phi^2 = O(\zeta_2)$ there is also a crossover, but to a linear or constant function times a logarithmic function of τ. The logarithmic function changes very slowly, and may be approximated roughly by a constant. The crossover occurs when

$$|\tau| \sim \zeta_2\log(\zeta_2|\tau|+\phi^2) \sim \zeta_2\log(\zeta_2)\ . \qquad (67)$$

We are thus led to make the following approximation

$$p(\phi) \sim \zeta_2\log\zeta_2, \quad \text{for} \quad \phi^2 = O(\zeta_2)\ . \qquad (68)$$

This estimate implies that when calculating the Fourier coefficients in (64) the integral should be cut off at $\phi \sim \sqrt{\zeta_2}$, giving the coefficients a $\frac{1}{2}\log(\zeta_2)$ dependence on ζ_2. It is now possible to balance powers in the matching Eq. (62), obtaining the order of magnitude relation (43).

V. SOLUTIONS FOR GENERAL VALUES OF ζ_H

For general values of ζ_h and d the differential equation (21), which has variable coefficients is not accessible to analytic techniques. In this section we present numerical solutions of the scaling exponents ζ_3 for arbitrary values of ζ_h, using a discretized version of the operator $\hat{B}_3$. Since the differential problem is a linear homogeneous equation with linear homogeneous boundary conditions, the discretized problem is also a homogeneous linear equation, implying that non-trivial solutions exist only when the determinant of the discretized operator vanishes. This determinant depends parametrically on ζ_3. Since the differential operator is defined on a compact domain we expect the determinant to vanish only for discrete values of ζ_3 for any given values of ζ_h and the dimensionality d. One solution is known to exist always, a constant zero-mode associated with $\zeta_3 = 0$. Our aim is to find the lowest lying positive real solutions ζ_3 for which the determinant vanishes.

FIG. 1. The scaling exponent ζ_3 as a functions of ζ_h found as the loci of zeros of the determinant of the matrix , for d=2.

The discretization of the operator $\hat{B}_3$ was carried out as follows: We defined a nine-point finite difference scheme for the second order equation (21). The discretization of the boundary conditions at $\rho = 1$ (25) is achieved using the same scheme, which requires in this case only three boundary points and one interior point, since on the boundary the radial derivative appears in first order. Using the symmetry of the problem we restricted the domain to one sixth of the circle, $0 < \phi < \pi/3$. The symmetry implies that original problem on the full circle is equivalent to the problem on the reduced domain with simple Neuman boundary conditions $\partial_\phi f(\rho,\phi) = 0.$ on the new boundary lines $\phi = 0, \pi/3$. As explained above, the discretized problem is a matrix eigenvalue problem $B_3\mathbf{\Psi} = 0$, where B_3 is a large sparse matrix, whose rank depends on the mesh of the discretization, and $\mathbf{\Psi}$ is the discretized version of the zero-mode f. We used NAG's sparse Gaussian elimination routines to find the zeros of $\det(B_3)$, and determined the values of ζ_3 for these zeros as a function of ζ_h. The results of this procedure for space dimensions $d = 2, 3, 4$ are presented in Figs. 1,2, and 3. The zero modes that correspond to any given value of ζ_3 can be found straightforwardly by inverse iterations of the matrix B_3 with an arbitrary initial vector.

A. Results

The various branches shown in Figs. 1–3 can be organized on the basis of the perturbation theory near $\zeta_h = 0$ which was presented in Section 2. At $\zeta_h = 0$ we identify the actual starting points of the branches with the analytic solutions for the lowest lying positive values of ζ_3, which are 4, 6, 8 etc. In addition, for $d = 2$ we observe the highest negative value which is -2. Measuring the slopes of the various branches at $\zeta_h = 0$ we find full agreement with the perturbative predictions.

FIG. 2. Same as Fig.1, but for $d = 3$.

FIG. 3. Same as Fig.1, but for $d = 4$.

We see that in all dimensions the branch which begins at $\zeta_h = 0$, $\zeta_3 = 4$ continues in the non-perturbative region without crossing any other branch until it ends at $\zeta_h = 2$, $\zeta_3 = 0$. This branch is a continuation of the

lowest lying positive branch predicted by the perturbation theory. The negative branch (shown only for $d = 2$) never rises above its perturbative limit and is not relevant for the scaling behavior at any value of ζ_h. Note also that the point $\zeta_h = 2$, $\zeta_3 = 0$ is an accumulation point of many branches, and we display only a part of the actual spectrum near this point. These branches seem to approach the accumulation point with a slope that grows without limit. This finding is in agreement with the analytic result of the perturbation calculation presented in Section 4.

VI. CONCLUSIONS

It is well known by now that there exists a disagreement between the scaling exponents ζ_4 and the higher order exponents ζ_n computed via perturbative approaches and the predictions of another approach based on the fully fused structure functions. The latter approach seems to be consistent with the results of numerical simulations in two spatial dimensions. The main conclusion of the present paper is that this disagreement cannot be ascribed to a formal failure of the perturbation theory. If we accept the statement that the scalar diffusivity is irrelevant, and compare the predictions of perturbation theory at both ends of the range of the allowed values of the parameter ζ_h, we find that they are in excellent agreement with the non-perturbative calculation of the scaling exponent, again subject to the assumption that the diffusivity is irrelevant. Therefore, if we want to understand the discrepancy between the two approaches mentioned above, there are a few possibilities that have to be sorted out by further research:

(i) The crucial assumption that goes to the fully fused approach, which is the linearity of the conditional average of the Laplacian of the scalar, is wrong.

(ii) The computation of the zero modes which is achieved by discarding the viscous terms in $\hat{B}_n$ is irrelevant for the physical solution. It is not impossible that the limits $\zeta_h \to 0$ and $\kappa \to 0$ do not commute, giving rise to some wicked properties of the very small ζ_h regime. That this is a possibility is underlined by recent calculations of a shell model of the Kraichnan model [13], in which it was shown that the addition of any minute diffusivity changes the nature of the zero modes qualitatively.

(iii) Lastly, and maybe most interestingly, it is possible that the physical solution is not strictly scale invariant through all the range of allowed distances [14]. In other words, it is possible that $\mathcal{F}_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ is not a homogeneous functions with a fixed homogeneity exponent ζ_3, but rather (for example), that ζ_3 depends on the ratios of the separations (or, in other words, the geometry of the triangle defined by the coordinates). If this were also the case for even correlation functions $\mathcal{F}_{2n}$, this would open an exciting route for further research to understand how non-scale invariant correlation functions turn, upon fusion, to scale invariant structure functions.

In light of the numerical results of Ref. [15] and the experimental results displayed in [16,17] we tend to doubt option (i). If we were to guess at this point we would opt for possibility (ii). More work however is needed to clarify this important issue beyond doubt.

APPENDIX A: THE COEFFICIENTS

$$
o_1: \quad \begin{array}{ll}
\{0,\ 0,\ 1,\ -1,\ 1,\ 8\} & \{0,\ 0,\ 2,\ -2,\ 0,\ -2\} \\
\{0,\ 0,\ 2,\ -1,\ 1,\ -4\} & \{0,\ 1,\ 0,\ -1,\ 0,\ -2\} \\
\{0,\ 1,\ 0,\ 0,\ 1,\ -12\} & \{0,\ 1,\ 0,\ 2,\ 1,\ 8\} \\
\{0,\ 2,\ 0,\ 0,\ 0,\ -2\} & \{0,\ 2,\ 0,\ 1,\ 1,\ -4\} \\
\{0,\ 2,\ 0,\ 2,\ 0,\ 2\} & \{0,\ 2,\ 0,\ 3,\ 1,\ 4\} \\
\{1,\ 0,\ 1,\ -1,\ 1,\ -8\} & \{1,\ 1,\ 0,\ 0,\ 1,\ 8\} \\
\{1,\ 1,\ 0,\ 2,\ 1,\ -8\} & \{2,\ 0,\ 0,\ 0,\ 0,\ -2\} \\
\{2,\ 0,\ 0,\ 1,\ 1,\ 4\} &
\end{array} \tag{A1}
$$

$$
o_2: \quad \begin{array}{ll}
\{0,\ 0,\ 1,\ -1,\ 1,\ -4\} & \{0,\ 1,\ 0,\ 0,\ 1,\ 4\} \\
\{0,\ 1,\ 0,\ 1,\ 0,\ 2\} & \{1,\ 0,\ 0,\ 0,\ 0,\ -2\}
\end{array} \tag{A2}
$$

$$
o_3: \quad \begin{array}{ll}
\{0,\ 0,\ 1,\ 0,\ 2,\ -4\} & \{0,\ 0,\ 1,\ -2,\ 2,\ 4\} \\
\{0,\ 0,\ 2,\ 0,\ 0,\ 1\} & \{0,\ 0,\ 2,\ -2,\ 0,\ -1\} \\
\{0,\ 0,\ 2,\ 0,\ 2,\ 2\} & \{0,\ 0,\ 2,\ -2,\ 2,\ -2\} \\
\{0,\ 1,\ 0,\ -1,\ 0,\ -1\} & \{0,\ 1,\ 0,\ 1,\ 0,\ 3\} \\
\{0,\ 1,\ 0,\ 3,\ 0,\ -2\} & \{0,\ 1,\ 0,\ 0,\ 1,\ -2\} \\
\{0,\ 1,\ 0,\ 2,\ 1,\ 2\} & \{0,\ 1,\ 0,\ -1,\ 2,\ -2\} \\
\{0,\ 1,\ 0,\ 1,\ 2,\ 2\} & \{0,\ 1,\ 1,\ 0,\ 1,\ 2\} \\
\{0,\ 1,\ 1,\ 2,\ 1,\ -2\} & \{0,\ 1,\ 1,\ -1,\ 2,\ -4\} \\
\{0,\ 1,\ 1,\ 1,\ 2,\ 4\} & \{0,\ 2,\ 0,\ 0,\ 0,\ -1\} \\
\{0,\ 2,\ 0,\ 2,\ 0,\ 2\} & \{0,\ 2,\ 0,\ 4,\ 0,\ -1\} \\
\{0,\ 2,\ 0,\ 1,\ 1,\ -2\} & \{0,\ 2,\ 0,\ 3,\ 1,\ 2\} \\
\{0,\ 2,\ 0,\ 0,\ 2,\ 2\} & \{0,\ 2,\ 0,\ 2,\ 2,\ -2\} \\
\{1,\ 0,\ 1,\ -1,\ 1,\ -2\} & \{1,\ 0,\ 1,\ 1,\ 1,\ 2\} \\
\{1,\ 1,\ 0,\ 1,\ 0,\ -2\} & \{1,\ 1,\ 0,\ 3,\ 0,\ 2\} \\
\{1,\ 1,\ 0,\ 0,\ 1,\ 2\} & \{1,\ 1,\ 0,\ 2,\ 1,\ -2\} \\
\{2,\ 0,\ 0,\ 0,\ 0,\ 1\} & \{2,\ 0,\ 0,\ 2,\ 0,\ -1\}
\end{array} . \tag{A3}
$$

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